European Journal of Combinatorics 30 (2009) 1387–1391

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Preface

The notion of an association scheme is at the center of a variety of mathematical concepts, most of which have been defined and investigated in order to understand groups from a more general point of view; or to mimic group theoretic ideas and techniques within a somewhat wider algebraic (or combinatorial) framework. Its simplicity, its appearance in so many different disguises, and its significance in so many different areas in mathematics have much to do with the wonderful balance between freedom and necessity that is characteristic of scheme theory.

Apart from presenting the individual articles of this special issue on association schemes in this preface we wish to briefly survey the most important of the above-mentioned concepts (all of which will be discussed in this issue) and to sketch their interrelationships. All concepts of algebraic nature have to do with the notion of a C-algebra, all concepts of combinatorial flavor with coherent configurations. We start on the algebraic side.

1. C-Algebras

An associative algebra A with identity element 1 over the complex number field is called a C-*algebra* if it possesses a finite (linear) basis B with real structure constants and $1 \in B$ and an involutory semilinear (complex) algebra antiautomorphism * with $B^* = B$ such that the following two conditions hold.

(i) The coefficient of 1 in a product of two elements c and d of B is 0 if $c \neq d^*$ and positive if $c = d^*$.
(ii) The linear functional on A mapping each element b of B to the coefficient of 1 in the product of b and b^* is a *-homomorphism.

Commutative C-algebras have been introduced by Y. Kawada in 1942; cf. [19]. Implicitly, however, commutative C-algebras appear already in articles by G. Hoheisel and (in the disguise of centralizer algebras of permutation groups) by I. Schur; cf. [18] and [22], respectively. A fairly complete picture of related algebraic structures is given in H. Blau's article in this issue.

The step from commutative C-algebras to C-algebras in general has been made in [11] by S. A. Evdokimov, I. N. Ponomarenko, and A. M. Vershik.

2. Table algebras

C-algebras in which the distinguished basis has only nonnegative (real) structure constants are called *table algebras*.

Similar to C-algebras, table algebras have mostly been investigated under the additional hypothesis of commutativity, and as such were they introduced by Z. Arad and H. Blau in [1]. The general theory has been considered more recently in [2,3,6].

0195-6698/$ – see front matter
doi:10.1016/j.ejc.2008.11.002

The center of the group algebra of a finite group over the complex number field together with the basis consisting of the conjugacy class sums is an example of a commutative table algebra. A second example is the algebra of the complex-valued class functions of a finite group (with respect to componentwise multiplication) together with the basis consisting of all irreducible complex characters. The notion of a commutative table algebra and the interest in these algebras arose mainly from the desire to consider these two concepts simultaneously.

3. Specific classes of table algebras

There are classes of table algebras which appear under different names in the literature. We mention here only the following two.

Commutative table algebras are called *fusion rule algebras* (or *fusion algebras*) if the structure constants of their distinguished basis B are integral and if the coefficient of 1 in b^*b is 1 for each element b in B; cf. [12].

Table algebras are called *hypergroups* if the algebra homomorphism in condition (ii) sends all elements of the distinguished basis to 1; cf. [20].

Besides these two specific classes of table algebras, there are quite a few classes of associative algebras that arise in other areas of mathematics and turn out to be table algebras as well. Examples are adjacency algebras of association schemes (scheme rings, Bose–Mesner algebras), Schur rings, centralizer algebras (centralizer rings), Hecke algebras, and group algebras (group rings) of finite groups. We shall meet these algebras now when we come to the combinatorial part of this introduction.

4. Coherent configurations

A *coherent configuration* on a nonempty finite set X is defined to be a partition S of the cartesian product $X \times X$ that satisfies the following three conditions.

(i) The set $\{(x, x) \mid x \in X\}$ is a union of elements of S.
(ii) For each element s in S, $\{(y, z) \mid (z, y) \in s\}$ is an element of S.
(iii) For any five elements p, q, r in S, and y, z in X, with $(y, z) \in r$, the number of elements x in X with $(y, x) \in p$ and $(x, z) \in q$ does not depend on y or z.

The notion of a coherent configuration has been introduced independently by B. Weisfeiler and A. A. Lehman and by D. G. Higman; cf. [23] and [15], respectively.

Examples of coherent configurations arise from group actions as follows. Let G be a group acting on a nonempty finite set X, and let S be the set of the orbits of the componentwise action of G on $X \times X$. Then S is a coherent configuration on X.

Coherent configurations that arise from a group action in the above described way are called *schurian*.

Coherent configurations give rise to associative algebras. In fact, let S be a coherent configuration on a (finite) set X, and let R denote the endomorphism ring of the $|X|$-dimensional complex vector space. Condition (iii) of a coherent configuration guarantees that the linear span of the adjacency endomorphisms of the elements of S is a subring of R. This subring is an associative algebra over the complex number field and is referred to as the *adjacency algebra* of S.

Adjacency algebras of coherent configurations can also be defined purely algebraically (without reference to coherent configurations). Higman did that in [16] and called these algebras *coherent algebras*. Weisfeiler and Lehman called their algebras *cellular algebras*; cf. [23]. Occasionally the term *cellular ring* is used instead of cellular algebra.

5. Association schemes

A coherent configuration S on a set X is called *homogeneous* if $\{(x, x) \mid x \in X\} \in S$. Nowadays, homogeneous coherent configurations are frequently called *association schemes* or just *schemes*; cf. [5, 24].

Schemes with commutative adjacency algebra are called *commutative*. Schemes all elements of which are symmetric (relations) are called *symmetric*. It is easy to see that symmetric schemes are commutative and that the converse does not hold.

Symmetric association schemes were introduced by R. C. Bose and T. Shimamoto in 1952; cf. [8]. Implicitly the notion appears already in [7]. About twenty years later, commutative association schemes became a research subject in themselves, when P. Delsarte considered them in his investigation of codes and designs; cf. [9].

Several authors include commutativity or even symmetricity in the definition of association schemes (thereby distinguishing between homogeneous coherent configurations and association schemes). This is particularly common when association schemes are considered in connection with graphs, codes, or statistical questions; cf., e.g., [4].

The adjacency algebra of an association scheme is also known as its *scheme ring* or its *Bose–Mesner* algebra. The latter term is often used when commutativity or symmetricity is included in the definition of association schemes.

It is obvious that adjacency algebras of association schemes are table algebras.

Recently algebras were introduced that correspond to coherent configurations in the same way as C-algebras correspond to homogeneous coherent configurations; cf. [10].

6. Specific classes of association schemes

Just as schemes, in the guise of their adjacency algebras, appear as table algebras, there are a number of mathematical objects that can be identified with specific classes of association schemes. (This can be considered as one of the reasons for the growing interest in scheme theory.)

On the algebraic side, there are, for instance, the finite groups and, more generally, the transitive group actions on finite sets, which can be viewed as schemes. Examples on the combinatorial side are the distance-regular graphs and, most notably, the finite buildings (which include, of course, all finite projective planes and all projective spaces over finite fields).

The schemes that correspond to these objects are (in the same order) the *thin* schemes (which are defined as schemes in which all elements are maps), the schurian schemes (which have been mentioned earlier), the P-polynomial schemes, and the so-called Coxeter schemes of finite valency; cf. [25]. The adjacency algebras of the thin schemes are the *group algebras* (*group rings*); the adjacency algebras of schurian schemes are often called *centralizer algebras* or *Hecke algebras*. Some authors also speak about *centralizer rings* or *Hecke rings*; cf., e.g., [5].

A *fusion* of a scheme S on a set X is a collection of unions of elements of S which again forms a scheme on X. Adjacency algebras of fusions of thin schemes are called *Schur rings* (or *S-rings*). Schur rings were introduced in [22].

There are more classes of mathematical objects that have their description in scheme theory, among them Moore geometries, symmetric and quasi-symmetric designs, partial geometries, and specific classes of difference sets.

At this final stage of our brief survey, one might ask whether the finiteness in the above-mentioned identification of finite groups with thin schemes, in the identification of transitive group actions on finite sets with schurian schemes, and in the identification of finite buildings with Coxeter schemes of finite valency is really essential. In fact, it is not. All three identifications go through also for infinite objects; one just has to drop the requirement of the finiteness of the underlying set in the definition of a scheme; cf. [24,25].

Another natural question arises. Do any of the algebras mentioned in this introduction need to be considered over only the complex number field? Here the answer is negative, too. Recent work of A. Hanaki, M. Hirasaka, and K. Uno shows that adjacency algebras over fields of positive characteristic yield interesting information about the structure of schemes of finite valency; cf., e.g., [14].

We conclude this preface with a few comments on the specifics of this issue and its purpose.

The examples mentioned as specific classes of association schemes show that association schemes are typically investigated from three different points of view: as algebras (table algebras, scheme rings), purely structure theoretically (Jordan–Hölder theory, Sylow theory), and as geometries (distance-regular graphs, designs, finite geometries, buildings). It seems that this trichotomy is a specific feature which schemes have inherited from groups. Groups have always been treated with a more algebraic emphasis (representation theory, character theory), structure theoretically (maximal subgroups, local group theory), and geometric-combinatorially (coset geometries, graphs). Each of these aspects contributes to the general understanding of groups, and the same seems to apply to schemes.

Depending on the emphasis that one puts on one or the other feature of association schemes, one finds a wealth of ideas and perspectives in scheme theory.

Algebraic features of association schemes are the subject of three articles in this issue. Akihide Hanaki surveys the theory of representations of association schemes which he models after the representation theory of finite groups. His emphasis is on modular representations. Harvey Blau's article illuminates the algebraic aspect of association schemes within the more abstract framework of table algebras. Schur rings, which can be viewed as the prototype of a table algebra, are surveyed in the contribution of Mikhail Muzychuk and Ilia Ponomarenko to this issue.

The development of abstract scheme theory has led to quite a few generalizations of group theoretic notions, techniques, and facts; cf. [13,17,21,25]. The lines of development in this direction are the theme of Paul–Hermann Zieschang's contribution to this issue. His article also discusses sufficient criteria for schemes to be schurian, the first step towards a fully developed extension theory. The more general question of how to measure schurity together with the question of how to characterize schemes with the help of complete sets of invariants lies at the heart of the article of Sergej Evdokimov and Ilia Ponomarenko on coherent configurations. Their contribution also deals with algorithmic applications of scheme theory.

The type of question one might wish to ask about geometric-combinatorial aspects of association schemes is, in a certain sense, foreshadowed by the overwhelming literature on distance-regular graphs as well as by the connection between buildings and schemes. Distance-regular graphs are the starting point of the contribution of Bill Martin and Hajime Tanaka to this issue. Their article records the impact of commutative association schemes on codes, designs, and linear and semi-definite programming. More recent and further reaching implications of scheme theory on spherical designs, programming, and optimization questions are addressed in the article of Eiichi and Etsuko Bannai within this issue. Their article also includes sphere packing and generalizations of spherical designs to compact symmetric spaces.

The survey character of the individual articles in this issue as well as their numerous hints and suggestions for further research prompted us to call this issue

Association schemes: ideas and perspectives

It is our hope that the various different approaches presented in this issue give an impression of how multifaceted the area around scheme theory is and in how many different ways schemes can be treated. We would be glad to see this issue being accepted by a broader audience as a stimulating introduction to the manifold aspects of the young and vibrant theory of association schemes.

Acknowledgement

The editors wish to thank Michel Deza for the suggestion and for his encouragement to edit this special issue on association schemes as an issue within the *European Journal of Combinatorics*.

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Available online 28 January 2009

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